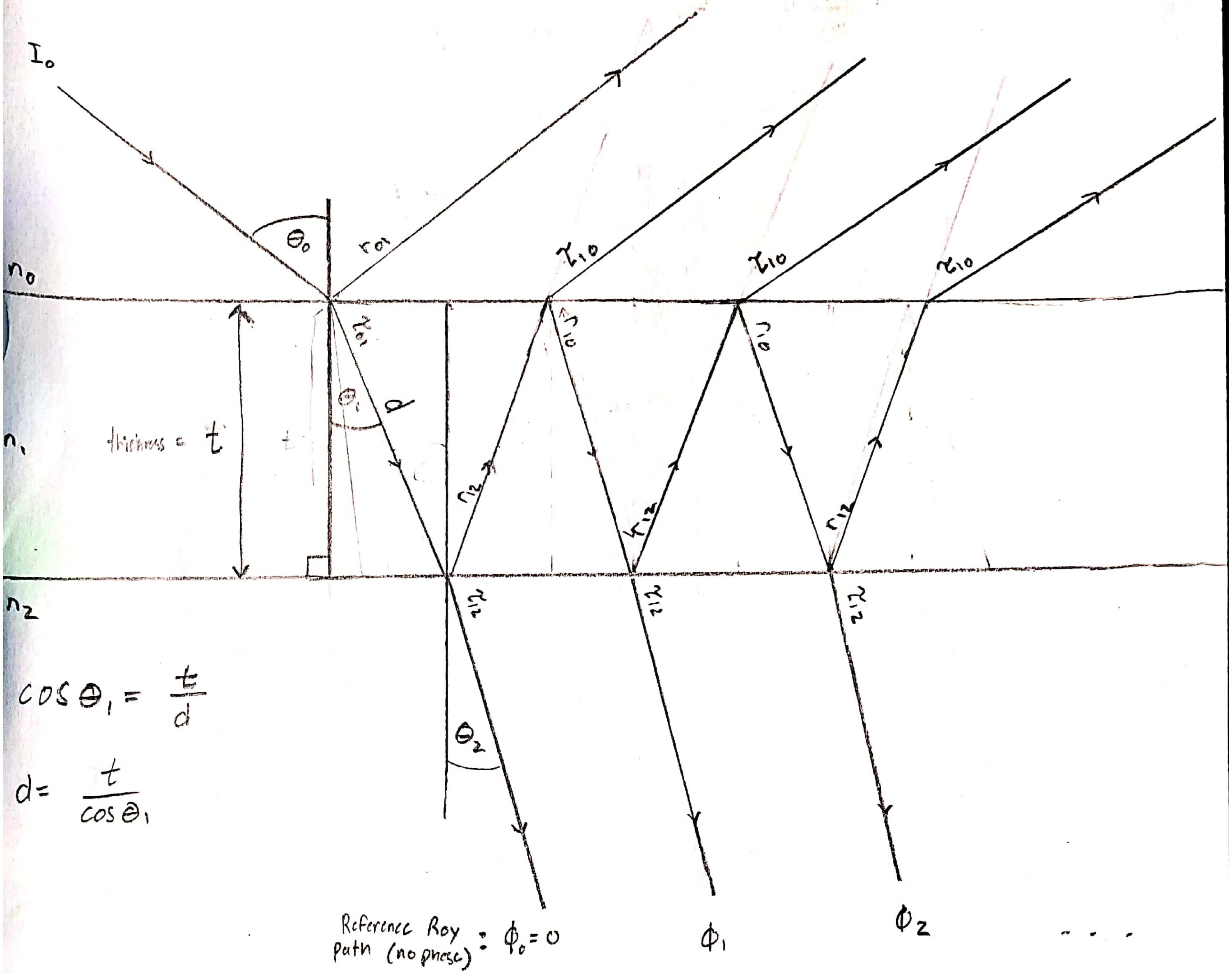
3-Layer Thin-Film Interference Derivation :



$$\cos\theta_1 = \frac{t}{d}$$

$$d = \frac{t}{\cos\theta_1}$$

We consider the energy carried by each outgoing ray leaving the thin film. The sum of the energy for each ray must equal the incident energy. The transmission and reflection coefficients τ_{ij} and r_{ij} represent the fractional energy that persists after interaction with the interface $i|j$. (example: oil | water). Note that $\tau_{i|j}$ and $r_{i|j}$ are the Fresnel coefficients and depend on the polarization of the incident light. We assume all polarizations are uniformly present in the incident radiation so we simply average the s and p polarization Fresnel coefficients.

The optical path length between transmitted rays (with respect to the reference path) is $2 \times d$. From this we can determine the phase offset for each transmission path as well as the fractional energy that each ray carries. By transmission path, we mean rays leaving through bottom medium n_2. Below, $\Delta_{i/j}$ denotes the π phase shift that occurs when a wave enters a medium that it travels slower in. $\Rightarrow$ $\Delta_{i/j} = \left\{ \begin{matrix} \pi & \text{if } n_j > n_i \\ 0 & \text{if } n_i \leq n_j \end{matrix} \right\}$

$$OPL_n = k\left(2d + \Delta_{1|0} + \Delta_{1|2}\right) \Rightarrow \frac{\phi_n}{2\pi} = \frac{OPL_k}{\lambda_1} = \frac{k}{\lambda_1}\left(2d + \Delta\right)$$

where $\Delta = \Delta_{1|0} + \Delta_{1|2}$

$$\Rightarrow \phi_n = k \cdot \frac{2\pi}{\lambda_1}\left(2d + \Delta\right)$$

Geometrically, we see that $d = \frac{t}{\cos\theta_1}$

$$\Rightarrow \phi_k = k \cdot \phi$$

$$\Rightarrow \phi_n = k \cdot \left[\frac{2\pi}{\lambda_1}\left(\frac{2t}{\cos\theta_1} + \Delta\right)\right]$$

λ_1 is the wavelength of light in medium n_1. Since $\lambda_i n_i = \frac{c}{f} = \text{const.}$ we have that $\lambda_1 = \lambda_0 \frac{n_0}{n_1}$.

Taking n_0 to be vacuum, we have that $n_0 = 1 \Rightarrow \lambda_1 = \frac{\lambda_0}{n_1}$

$$\Rightarrow \boxed{\phi = \frac{2\pi n_1}{\lambda_0}\left(\frac{2t}{\cos\theta_1} + \Delta\right)}$$

Note, this is different from the expression on generalized.net. The cosine is inverted.

The fraction of energy carried by the k^{th} transmission path associated with ϕ_n can be found simply by tracing the path and serially taking the product of the denoted Fresnel coefficients. Note that after the reference path, all subsequent transmission paths require additional bounces inside medium n_1 that come in pairs (one off the bottom interface $n_1|n_2$ and one off the top interface $n_1|n_0$).

Hence for the reference path $k=0$, the transmission factor is $\tau_{01}\tau_{12}$. For the $k=1$ path — // — $\tau_{01} r_{12} r_{10} \tau_{12}$. For $k=2$ path — // — $\tau r_{12} r_{10} r_{12} r_{10} \tau_{12}$. We can see a pattern here that generalizes more completely to the following:

Transmission Coeff for k^{th} path $\longrightarrow$ $(\tau_{01}\tau_{12})(r_{12}r_{10})^k$

putting all of this together, we have that the transmitted intensity (Complex Amplitude Squared) is both dependent on the Fresnel factors as well as the interference induced by phase effects from the reference path. Hence,

$$I_T = I_0 \cdot T = I_0 \cdot \left| \sum_{k=0}^{\infty} (\tau_{01}\tau_{12})(r_{12}r_{10}e^{i\phi})^k \right|^2$$

$$T = |\tau_{01}\tau_{12}|^2 \left| \sum_{k=0}^{\infty} (r_{12}r_{10}e^{i\phi})^k \right|^2 \longleftarrow$$

This is an infinite geometric series that can be written analytically.

$$= |\tau_{01}\tau_{12}|^2 \left| \frac{1}{1-(r_{12}r_{10}e^{i\phi})} \right|^2$$

Let $(\tau_{01}\tau_{12}) = \beta \in \mathbb{R}$
and $(r_{12}r_{10}) = \alpha \in \mathbb{R}$

$$= \frac{\beta^2}{(1-(r_{12}r_{10})e^{i\phi})(1-\overline{(r_{12}r_{10})}e^{-i\phi})} = \frac{\beta^2}{1-\alpha(e^{i\phi}+e^{-i\phi})+\alpha^2}$$

$$T = \frac{\beta^2}{1-2\alpha\cos(\phi)+\alpha^2}$$

For completeness, we have

$$\phi = \frac{2\pi n_1}{\lambda_0}\left(\frac{2t}{\cos\theta_1} + \Delta\right)$$

$$T = \frac{\beta^2}{\alpha^2 - 2\alpha\cos\phi + 1}$$

$$R = 1 - T$$

Using Snell's Law we can write $\cos\theta_1$ in terms of the incident angle. θ_0.

$n_0 \sin\theta_0 = n_1 \sin\theta_1$ ← Snell's Law

$$\sin\theta_1 = \frac{n_0}{n_1}\sin\theta_0$$

$$\cos\theta_1 = \sqrt{1 - \sin^2\theta_1}$$

$$\cos\theta_1 = \sqrt{1 - \left(\frac{n_0}{n_1}\sin\theta_0\right)^2}$$

Note: Total Internal reflection occurs when $1 - \left(\frac{n_0}{n_1}\sin\theta_0\right)^2 < 1$